Self-focusing and solitonlike structures in materials with competing quadratic and cubic nonlinearities

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We study the mutual influence of quadratic and cubic nonlinearities on the propagation of the coupled fundamental and second harmonic waves in asymmetric optical media. For attractive potentials with positive coupling parameters, it is shown that, in systems with two and three transverse dimensions, mutually trapped waves can self-focus until collapse whenever their respective powers exceed some thresholds. On the contrary, coupled waves diffracting in a one-dimensional plane never collapse and may evolve towards stable solitonlike structures. For higher transverse dimension numbers, we investigate the question of forming two-component solitons and determine criteria for their stability. [S1063-651X(97)13502-4]

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I. INTRODUCTION

The possibility of generating stable solitonlike beams in media with a purely quadratic or so-called $\chi^{(2)}$ nonlinearity has been the subject of many recent investigations, both theoretical [1–9] and experimental [10–14]. These solitary waves consist of two components, a fundamental wave and its second harmonic, coupled together through cascaded nonlinear three-wave interaction processes that usually take place in asymmetric media privileging second harmonic generation. It has recently been shown analytically that stable two-component solitary waves do exist for all dimension numbers D of physical interest [3,7–9], and that, regardless of the initial wave functions, a catastrophic collapse should never occur in purely quadratic materials [8]. Here and in the following, the letter D refers to the space dimension number of the transverse plane, perpendicular to the direction of propagation. As will be seen further, dealing with three-dimensional (3D) waves will mean that the propagation equations will account not only for the transverse diffraction of the wave envelopes, but also for their variations in time.

However, even in materials where the $\chi^{(2)}$ nonlinearity may be dominant, there always exists a cubic or so-called $\chi^{(3)}$ nonlinearity. One of the effects of this $\chi^{(3)}$ nonlinearity is that the refractive index becomes dependent on the intensity, which is also known as the Kerr effect. In media with a purely cubic nonlinearity, this Kerr effect leads to a quite different evolution of dispersive, weakly nonlinear wave packets, as described by the well-known nonlinear Schrödinger (NLS) equation: although (one-component) solitons exist and are stable in the one-dimensional case $D=1$ for which the NLS equation is integrable [15], such localized beams become unstable in two or more dimensions where they either disperse or self-focus until a destructive collapse takes place at a finite propagation distance (see, e.g., Ref. [16]).

In the light of these features, an important question concerning the propagation of localized coupled waves in materials with both $\chi^{(2)}$ and $\chi^{(3)}$ nonlinearities is thus to determine whether they converge towards stable solitary waves, or simply spread out, or self-focus until a collapse at a finite propagation distance. The dynamical equations for coupled waves in media supporting an interplay between these two nonlinearities have been derived in [17], and preliminary investigations showed that two-component solitary waves may exist and be stable in the simplest one-dimensional situation, $D=1$ [18–20]. Generally, the evolution of the coupled waves depends on their initial (or incident) values and on the transverse dimension characterizing the nonlinear equations that govern their propagation.

In this paper, we investigate the influence of both kinds of nonlinearities (quadratic and cubic) on the possible formation of stable solitary waves and the collapse of unstable coupled waves. The model equations are presented in Sec. II. After deriving the main invariants and the characteristic relation governing the mean square radius attached to the fundamental and second harmonic waves in Sec. III, we determine some rigorous criteria, for or against the collapse of those mutually trapped waves, depending on their incident data at $z=0$. We detail this question for the pure $\chi^{(2)}$ nonlinearity and for the pure $\chi^{(3)}$ one, separately, then for the combined problem in Sec. IV. In Sec. V, we analyze the possibility of realizing stable solitonlike structures (so-called combined or C-type solitons following the terminology introduced in [18]) for all transverse dimensions of physical relevance $D\leqslant 3$. Finally, in Secs. VI and VII, we present variational and numerical calculations supporting our theoretical results. They confirm, in particular, that waves with a

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sufficient power can collapse under the combined influence of $\chi^{(2)}$ and $\chi^{(3)}$ nonlinearities.

II. THE PROPAGATION EQUATIONS

The dynamical equations for 1D wave packets in media with both $\chi^{(2)}$ and $\chi^{(3)}$ nonlinearities were recently derived in [17]. The transverse profile of the field was taken into account, and shown to alter the linear dispersion relation and the strength of the nonlinearity through certain structure coefficients. We consider the straightforward extension of these equations to D dimensions:

$$i\partial_z w + s\vec{\nabla}_\perp^2 w + w^* v + |w|^2 w + \rho|v|^2 w = 0, \quad (1)$$

$$2i\partial_z v + s\vec{\nabla}_\perp^2 v - \beta v + \tfrac{1}{2}w^2 + \lambda|v|^2 v + \rho|w|^2 v = 0, \quad (2)$$

where the symbol $*$ denotes the complex conjugate function. Written in a convenient normalized form, Eqs. (1) and (2) are valid when the material is lossless and as long as the fundamental frequency ω_1 and its second harmonic $\omega_2 = 2\omega_1$ are far from any internal material resonance. The slowly varying complex envelope functions of the fundamental field, $w(\vec{r}_\perp, z)$, and of the second harmonic, $v(\vec{r}_\perp, z)$, are here assumed to propagate with a constant polarization along the z axis. The transverse Laplacian refers to a general transverse plane of dimension $D \leqslant 3$, spanned by the radial vector $\vec{r}_\perp$. The spatial coordinates (x,y) constitute two of the three dimensions and correspond to the usual diffraction, while the third one corresponds to a retarded time, with respect of which the variations of the wave envelopes can account for the group-velocity dispersion [21]. Note that in comparison with the equations given in [17], the notation has been changed by a simple transformation.

We want to study the solutions to Eqs. (1) and (2) in arbitrary dimensions $D \leqslant 3$, keeping the linear dispersion relation and the definition of the real parameters s, β, λ, and ρ unchanged. Therefore, we ignore any effect of the transverse profile of the field on the linear dispersion relation and on these parameters. In this case, the quantities s, β, ρ, and λ are given by [17]

$$s = \mathrm{sgn}\{\tilde{\chi}_{1s}^{(3)}\}, \quad \beta = \frac{3}{4}\left[\frac{\tilde{\chi}_{1s}^{(3)}/\chi_3}{(\tilde{\chi}_1^{(2)}/\chi_2)^2}\right]\Delta\beta,$$

$$\lambda = \frac{16\tilde{\chi}_{2s}^{(3)}}{\tilde{\chi}_{1s}^{(3)}}, \quad \rho = \frac{8\tilde{\chi}_{1c}^{(3)}}{\tilde{\chi}_{1s}^{(3)}}, \quad (3)$$

where $\Delta\beta = (2k_1 - k_2)/k_1$ is the phase mismatch and $\chi_2 \equiv n_1\sqrt{\epsilon_0\chi_3}$ and χ_3 are given normalization parameters, with ϵ_0 being the vacuum permittivity. The wave number k_p is related to the frequency ω_p by the linear dispersion relation

$$k_p = n_p\omega_p/c, \quad n_p^2 = 1 + \mathrm{Re}\{\tilde{\chi}_p^{(1)}\}, \quad (4)$$

where n_p is the index of refraction associated with the pth wave and c the speed of light in vacuum. The scalars $\tilde{\chi}_p^{(j)} \equiv \tilde{\chi}^{(j)}(\omega_p)$ denote the Fourier components at frequency ω_p of the jth order susceptibility tensor [17]. Thus, $\tilde{\chi}_1^{(2)} = \tilde{\chi}_2^{(2)}$ represents the quadratic nonlinearity, $\tilde{\chi}_{ps}^{(3)}$ the part of the cubic nonlinearity that is responsible for the self-phase modulation, and $\tilde{\chi}_{1c}^{(3)} = \tilde{\chi}_{2c}^{(3)}$ the part of the cubic nonlinearity that is responsible for the cross-phase modulation.

Equations (1) and (2) are derived using a perturbation expansion, in which the normalized amplitude of the field plays the role of the small order parameter $\epsilon \ll 1$, in accordance with the assumption of weak nonlinearity [17]. The validity of Eqs. (1) and (2) requires that both the normalized scalar Fourier components of the second-order susceptibility tensor and the phase mismatch are small as defined below:

$$|\tilde{\chi}_1^{(2)}/\chi_2| \sim \epsilon, \quad |\Delta\beta| \sim \epsilon^2, \quad (5)$$

while both the ratios $|\tilde{\chi}_{ps}^{(3)}/\chi_3|$ and $|\tilde{\chi}_{pc}^{(3)}/\chi_3|$ are of order unity. The normalization parameter χ_3 is chosen as $\chi_3 = |\tilde{\chi}_{1s}^{(3)}|$, which then fixes $\chi_2 = n_1\sqrt{\epsilon_0\chi_3}$. This scaling makes the parameter β, which represents the relative strength of the quadratic and cubic nonlinearity, be of order unity. Whenever one of the dimensions under consideration is associated with the retarded time, the special symmetry, assumed when writing the equations in the form of Eqs. (1) and (2), requires that the group-velocity mismatch $\Delta\beta' = (k_1' - k_2')/(|k_1''|\omega_1)$ and the group-velocity dispersion fulfill the requirements

$$|\Delta\beta'| \sim \epsilon^2, \quad k_1 k_1'' = 2k_1 k_2'' < 0, \quad (6)$$

with $k_p' \equiv \partial k_p/\partial\omega_p$ and $k_p'' \equiv \partial^2 k_p/\partial\omega_p^2$. This thereby implies that not only the phase mismatch, but also the group-velocity mismatch, must be small.

Among the various parameters defined in Eq. (3), we shall henceforth consider positive values for the dispersion coefficient and therefore set $s = +1$: in the 3D case with two transverse spatial coordinates and a retarded time variable, this can formally model *anomalous* group-velocity dispersion [21].

III. CONSERVED INTEGRALS AND THE VIRIAL IDENTITY

We begin by establishing the main integrals and relations attached to Eqs. (1) and (2), whose z-dependent solutions w, v and their derivatives are assumed to be localized in the transverse space with a sufficient algebraic decrease (for instance, w and v can be supposed to belong to the Sobolev space W_2^1, at least locally in z).

First, setting $s = +1$, we refind the so-called ''power'' (or ''mass'') integral by multiplying Eqs. (1) and (2) by w^* and v^*, respectively, which yields, after taking the imaginary part of the results,

$$\partial_z|w|^2 = -2\vec{\nabla}_\perp \cdot \mathrm{Im}(w^*\vec{\nabla}_\perp w) - 2\,\mathrm{Im}[(w^*)^2 v], \quad (7)$$

$$2\partial_z|v|^2 = -2\vec{\nabla}_\perp \cdot \mathrm{Im}(v^*\vec{\nabla}_\perp v) + \mathrm{Im}[(w^*)^2 v]. \quad (8)$$

We multiply Eq. (8) by 2 and integrate the sum of the resulting equations over the transverse space to get the integral

$$N = \|w\|_2^2 + 4\|v\|_2^2 \equiv N_w + 4N_v, \quad (9)$$

which remains preserved along the z axis. For notational convenience, we have made use of the standard L^p norms

$$\|f\|_p \equiv \left(\int |f|^p d\vec{r}_\perp \right)^{1/p}.$$

Besides, we multiply Eqs. (1) and (2) by w_z^* and v_z^* and select the real part of the sum of the space-integrated results to obtain the conserved Hamiltonian:

$$H \equiv (\|\vec{\nabla}_\perp w\|_2^2 + \|\vec{\nabla}_\perp v\|_2^2) + \beta\|v\|_2^2 - \text{Re} \int (w^2 v^*) d\vec{r}_\perp - \frac{1}{2}\|w\|_4^4 - \frac{\lambda}{2}\|v\|_4^4 - \rho\|vw\|_2^2. \quad (10)$$

We now derive a so-called virial relation in the form $\partial_z^2 I(z) = F(z)$ for localized solutions to Eqs. (1) and (2). Defined up to a normalization factor N, $I(z)$ corresponds to the mean square radius associated with both coupled waves, namely,

$$I(z) = \|\vec{r}_\perp w\|_2^2 + 4\|\vec{r}_\perp v\|_2^2 \equiv I_w(z) + 4I_v(z).$$

We decompose the calculations into two steps:

(i) We first multiply Eq. (1) by $(r_\perp^2 w^*)$ and Eq. (2) by $(2r_\perp^2 v^*)$ and sum up the imaginary part of the space-integrated results to get

$$\partial_z I(z) = 4 \text{ Im} \int \vec{r}_\perp \cdot (w^* \vec{\nabla}_\perp w + 2v^* \vec{\nabla}_\perp v) d\vec{r}_\perp. \quad (11)$$

(ii) Then we multiply Eq. (1) by $(\vec{r}_\perp \cdot \vec{\nabla}_\perp w^*)$ and integrate the real part of the result to obtain after a few integrations by parts:

$$\partial_z \text{Im} \int (\vec{r}_\perp \cdot \vec{\nabla}_\perp w) w^* d\vec{r}_\perp = 2\|\vec{\nabla}_\perp w\|_2^2 - \frac{D}{2}\|w\|_4^4 - D\rho\|wv\|_2^2 + \text{Re} \int w^2 \vec{r}_\perp \cdot \vec{\nabla}_\perp v^* d\vec{r}_\perp - \rho \int |v|^2 \vec{r}_\perp \cdot \vec{\nabla}_\perp |w|^2 d\vec{r}_\perp. \quad (12)$$

Next, repeating the same procedure on Eq. (2), we multiply the latter by $(\vec{r}_\perp \cdot \vec{\nabla}_\perp v^*)$ to find

$$2\partial_z \text{Im} \int (\vec{r}_\perp \cdot \vec{\nabla}_\perp v) v^* d\vec{r}_\perp = 2\|\vec{\nabla}_\perp v\|_2^2 - \frac{D}{2} \text{Re} \int (w^2 v^*) d\vec{r}_\perp - \frac{\lambda D}{2}\|v\|_4^4 - \text{Re} \int w^2 (\vec{r}_\perp \cdot \vec{\nabla}_\perp v^*) d\vec{r}_\perp + \rho \int |v|^2 \vec{r}_\perp \cdot \vec{\nabla}_\perp |w|^2 d\vec{r}_\perp. \quad (13)$$

Combining Eqs. (12) and (13) into the z derivative of Eq. (11) finally yields the virial identity

$$\partial_z^2 I(z) = 8(\|\vec{\nabla}_\perp w\|_2^2 + \|\vec{\nabla}_\perp v\|_2^2) - 2D\left\{ \text{Re} \int (w^2 v^*) d\vec{r}_\perp + \|w\|_4^4 + \lambda\|v\|_4^4 + 2\rho\|wv\|_2^2 \right\}, \quad (14)$$

which can also be written, using the definition (10), in the alternative forms

$$\partial_z^2 I(z) = 8(H - \beta\|v\|_2^2) + 2(4-D)\text{Re} \int (w^2 v^*) d\vec{r}_\perp + (4-2D)(\|w\|_4^4 + \lambda\|v\|_4^4 + 2\rho\|wv\|_2^2), \quad (15)$$

or, equivalently,

$$\begin{aligned}\partial_z^2 I(z) &= 2(4-D)(\|\vec{\nabla}_\perp w\|_2^2 + \|\vec{\nabla}_\perp v\|_2^2) + 2D(H - \beta\|v\|_2^2) - D(\|w\|_4^4 + \lambda\|v\|_4^4 + 2\rho\|wv\|_2^2) \\ &= 4(2-D)(\|\vec{\nabla}_\perp w\|_2^2 + \|\vec{\nabla}_\perp v\|_2^2) + 4D(H - \beta\|v\|_2^2) + 2D \text{ Re} \int (w^2 v^*) d\vec{r}_\perp. \end{aligned} \quad (16)$$

From expression (15), we recognize the typical functional $(H - \beta\|v\|_2^2)$ and the coefficient $(4-D)$ that characterize waves propagating in a purely $\chi^{(2)}$ material [8]. We also notice the dimensional coefficient $(4-2D)$ in front of the nonlinear potentials usually associated with the cubic NLS equation for w [16], which is easily recovered by setting $v=0$ in Eq. (1) and by disregarding Eq. (2).

IV. CRITERIA FOR WAVE COLLAPSES

In what follows, we derive analytical criteria for the existence or absence of wave collapse for transverse dimension numbers of physical relevance $D \leqslant 3$. We divide the problem into three parts and first discuss the simpler cases of media that have either a purely quadratic nonlinearity or a purely cubic one, respectively, before going through the more complicated problem for which both kinds of nonlinearities are present in Eqs. (1) and (2).

A. Absence of collapse in media with a ''pure'' $\chi^{(2)}$ nonlinearity

We first prove the absence of collapse for trapped waves propagating in a purely quadratic medium. Some arguments displaying this property were already given in [8]. Here, we establish a rigorous proof that the $\chi^{(2)}$ nonlinearities alone cannot promote a collapse *for any initial data*. To this aim, we omit the cubic contributions in Eqs. (1) and (2) and we bound from below the resulting virial relation (16) as follows:

$$\begin{aligned}\partial_z^2 I(z) &= 2(4-D)(\|\vec{\nabla}_\perp w\|_2^2 + \|\vec{\nabla}_\perp v\|_2^2) + 2D(H - \beta\|v\|_2^2) \\ &\geqslant \frac{(4-D)}{2}(\|\vec{\nabla}_\perp w\|_2^2 + 4\|\vec{\nabla}_\perp v\|_2^2) + 2D(H - \beta\|v\|_2^2). \end{aligned} \quad (17)$$

Then, using the well-known inequality (see, e.g., [16])

$$\|g\|_2^4 \leq \frac{4}{D^2} \|\vec{\nabla}_\perp g\|_2^2 \|\vec{r}_\perp g\|_2^2 \tag{18}$$

applied to any L^2-integrable function g, we construct the estimate

$$(N_w + 4N_v)^2 \leq \frac{4}{D^2} (\|\vec{\nabla}_\perp w\|_2^2 + 4\|\vec{\nabla}_\perp v\|_2^2) I(z), \tag{19}$$

which simply results from employing the obvious inequality

$$2ab \leq qa^2 + b^2/q \quad (q>0) \tag{20}$$

with $q=1$ to the cross-product of the expression $(\|\vec{\nabla}_\perp w\|_2 \|\vec{r}_\perp w\|_2 + 4\|\vec{\nabla}_\perp v\|_2 \|\vec{r}_\perp v\|_2)^2$. Inserting Eq. (19) into Eq. (17), we get

$$\partial_z^2 I(z) \geq \frac{(4-D)}{8} \frac{D^2 N^2}{I(z)} + C, \tag{21}$$

where C denotes the constant: (i) $C = 2D(H - \beta N/4)$ if $\beta>0$; (ii) $C = 2DH$ if $\beta \leq 0$.

Let us now suppose that a collapse may occur at a given distance z_c, in the usual sense $I(z) \to 0$ as $z \to z_c$, similarly to the singular solutions of the NLS equation [16]. Then, necessarily, $I(z)$ must decrease from a certain distance z_0 until vanishing at $z_c > z_0$. Therefore, we integrate Eq. (21) from $z_0 \leq z$ after multiplying it by $\partial_z I(z) < 0$ and obtain

$$E(z) \equiv [\partial_z I(z)]^2 + \frac{4-D}{4} D^2 N^2 \ln \frac{1}{I(z)} - 2CI(z) \leq E(z_0) < +\infty. \tag{22}$$

Since $E(z_0)$ is finite, Eq. (22) shows that the limit $I(z) \to 0$ is impossible for $D \leq 3$, which proves that in media with quadratic nonlinearities, a wave collapse cannot occur. However valid the initial data (incident waves) may be, this proof is a generalization, and thus an improvement, of the analysis performed in Ref. [8].

From the previous result, we conclude that the total mean square radius attached to the two waves, $I(z) = I_w(z) + 4I_v(z)$, where $I_w(z)$ and $I_v(z)$ are both positive, will never tend to zero. Note that, strictly speaking, this property $I(z) \not\to 0$ does not prevent one of the two virial components, $I_w(z)$ or $I_v(z)$, from vanishing while the other component could possibly keep a finite nonzero value. As a consequence, this would imply the gradient norm associated with the vanishing virial component to blow up (or to diverge) as $z \to z_c$, by virtue of the inequality (18). Even though such a situation where only one wave component blows up with a diverging gradient norm might lead to serious inconsistencies in, e.g., the conservation of the Hamiltonian integral, it is not forbidden *a priori*. In this sense, proving the absence of collapse from the nonvanishing of the virial integral $I(z)$ constitutes a weaker result than proving that the gradient norms remain bounded from above, as is well known in the simpler context of the NLS equation [16]. In view of the inequalities (18) and (19), the boundedness of gradient norms, from which the global existence of solutions follows, indeed assures that the corresponding virial integrals will never vanish for a given nonzero L^2 norm, whereas the reciprocal implication is not true in general.

On the contrary, as will be seen below, proving that a collapse occurs in the standard NLS sense $I(z) \to 0$ as $z \to z_c$ will imply by itself the simultaneous vanishing of the virial components $I_w(z)$ and $I_v(z)$, leading to the blow up of both gradient norms at the maximum propagation distance z_c, by virtue of the inequality (18) applied to both waves w and v, separately. Even if collapse should take place at a distance shorter than z_c, one can still expect in this situation that the divergence of one given solution would cause the divergence of the other one through their mutual nonlinear coupling, in such a way that the ultimate collapse distance should be identical for the two wave components.

B. Collapse in media with a ''pure'' $\chi^{(3)}$ nonlinearity

Investigating now the purely cubic case, we omit the $\chi^{(2)}$ contributions; i.e., we set $w^* v = w^2/2 = 0$ for the nonlinear terms of Eqs. (1) and (2). The present investigation is made in view of enlightening the coming more complicated $\chi^{(2)} - \chi^{(3)}$ problem; details exclusively attached to a purely cubic medium are planned to be presented in a forthcoming publication [23]. For a pure $\chi^{(3)}$ nonlinearity, we can first notice that the individual powers N_w and N_v are separate constants of motion (see, e.g., [22]), unlike the combined $\chi^{(2)} - \chi^{(3)}$ problem for which only the total power $N = N_w + 4N_v$ is conserved. Furthermore, the mismatch parameter β simply reduces to zero in that case; in this respect, any linear term such as this β contribution could easily be removed by a simple phase transformation $v \to v e^{-i\beta z/2}$. When taking all these properties into account, the Hamiltonian integral (10) reduces to

$$H^{(3)} = (\|\vec{\nabla}_\perp w\|_2^2 + \|\vec{\nabla}_\perp v\|_2^2) - \frac{1}{2} \|w\|_4^4 - \frac{\lambda}{2} \|v\|_4^4 - \rho \|vw\|_2^2, \tag{23}$$

and the virial identity (15) becomes

$$\partial_z^2 I(z) = 8H^{(3)} + (4-2D)(\|w\|_4^4 + \lambda \|v\|_4^4 + 2\rho \|wv\|_2^2). \tag{24}$$

1. Absence of collapse for D=1

We demonstrate that the energy integral $H^{(3)}$ can be bounded from below for low transverse dimension numbers, in order to show the global existence of the solutions w and v in the case $D=1$. For this purpose, we first recall the Sobolev inequality

$$\|f\|_4^4 \leq C_f \|\vec{\nabla}_\perp f\|_2^D \|f\|_2^{4-D} \tag{25}$$

valid for any L^p integrable function f with C_f=const. Applied to the solutions w, v in the 2D case ($D=2$), one has $C_w = 2/N_{c,w}$ and $C_v = 2/N_{c,v}$ where the best constants $N_{c,w}^{\text{best}}$ and $N_{c,v}^{\text{best}}$ optimizing Eq. (25) can be determined from the NLS ground-state equation [24]. Both these values are identically equal to the critical threshold N_c for a 2D self-focusing. Computed from the radially symmetric ground state of the cubic NLS, N_c has the precise value $N_c = 11.68$. Furthermore, we need the Schwarz inequality, together with Eq. (20) defined for $q=1$, to estimate

$$\|wv\|_2^2 \leqslant \|w\|_4^2 \|v\|_4^2 \leqslant \tfrac{1}{2}(\|w\|_4^4 + \|v\|_4^4). \tag{26}$$

Applying Eqs. (25) and (26) to the integral $H^{(3)}$ enables us to bound the latter as follows:

$$H^{(3)} \geqslant \mathcal{F}(\|\vec{\nabla}_\perp w\|_2, \|\vec{\nabla}_\perp v\|_2)$$
$$\equiv \|\vec{\nabla}_\perp w\|_2^2 \left(1 - \epsilon_w \frac{C_w}{2} N_w^{2-D/2} \|\vec{\nabla}_\perp w\|_2^{D-2}\right)$$
$$+ \|\vec{\nabla}_\perp v\|_2^2 \left(1 - \epsilon_v \frac{C_v}{2} N_v^{2-D/2} \|\vec{\nabla}_\perp v\|_2^{D-2}\right), \tag{27}$$

where (i) $\epsilon_w = (\rho+1)$ if $\rho>0$ or $\epsilon_w = 1$ if $\rho<0$, (ii) $\epsilon_v = \max\{\rho+\lambda, \rho\}$ if $\rho>0$ or $\epsilon_v = \max\{\lambda, 0\}$ if $\rho<0$ [taking the maximum value among the two previous ones depends whether λ is positive or negative].

Whatever the parameters (ρ,λ) may be, the constants ϵ_w and ϵ_v are non-negative. From the estimation (27), we deduce that $\mathcal{F}$ is bounded in the plane $(\|\vec{\nabla}_\perp w\|_2, \|\vec{\nabla}_\perp v\|_2)$ for subcritical dimensions $D<2$. So, the functional $H^{(3)} \geqslant \mathcal{F}$ remains bounded in turn from below for $D<2$, which indicates that in this case, fixed-point solutions, such as soliton-type structures, can be stable if they realize the minimum of the Hamiltonian (see the next section for a detailed discussion of soliton stability in the context of the full $\chi^{(2)}-\chi^{(3)}$ problem). In addition, recalling that a collapse characterized by $I(z) = I_w(z) + 4I_v(z) \to 0$ implies the divergence of the gradient norms $\|\vec{\nabla}_\perp w\|_2^2$ and $\|\vec{\nabla}_\perp v\|_2^2$, in accordance with the inequality (19), we easily see from Eq. (27) that such a collapse can never occur for $D<2$, since the finiteness of $H^{(3)}$, which is a constant of motion, implies that the two gradient norms must remain bounded in that case. In connection with this result, the nonvanishing of $I(z)$ can easily be recovered by applying the virial integration technique detailed in Sec. IV A. We now investigate the complementary situations $D \geqslant 2$.

2. Occurrence of collapse for $D \geqslant 2$

We first study the most natural configuration for having collapse, namely, the one corresponding to attractive potentials with positive coupling constants (nonlinearity coefficients) $\lambda,\rho>0$. From the virial identity (24), it merely follows that collapse must occur for $D \geqslant 2$ under the *sufficient* condition

$$H^{(3)} < 0. \tag{28}$$

Once this condition is fulfilled, both waves blow up at a maximum propagation distance z_c since the vanishing of $I(z)$ implies the simultaneous vanishing of the individual mean square radius $I_w(z)$ and $I_v(z)$. As previously recalled, even if collapse takes place at a shorter distance, the divergence of one given solution should force the divergence of the other one through their mutual coupling, so that the collapse distance should be the same for both wave components. Note that for $D>2$, the functional $\mathcal{F}$ is unbounded from below with $\epsilon_w = \rho+1$ and $\epsilon_v = \rho+\lambda$, which may indicate the presence of collapse (with the restriction, however, that an unbounded $\mathcal{F}$ does not necessarily imply an unbounded Hamiltonian).

Investigating the opposite case of a repulsive potential with $\rho<0$ for $D \geqslant 2$, we can use the virial relation (24) again and bound it from above:

$$\partial_z^2 I(z) \leqslant 8H^{(3)} + (4-2D)[(\rho+1)\|w\|_4^4 + (\rho+\lambda)\|v\|_4^4],$$

by means of the inequalities (26) and $(1-D/2)\rho \geqslant 0$. From this, we conclude that collapse can still occur for a negative coupling constant ρ under the requirement (28), provided that this parameter belongs to the narrow range of values: $\max\{-\lambda,-1\} < \rho < 0$.

Besides, at the critical dimension $D=2$, when using Eqs. (25) and (26) the quantity $H^{(3)}$ is found to satisfy for positive coupling parameters:

$$H^{(3)} \geqslant \|\vec{\nabla}_\perp w\|_2^2 \left[1 - (\rho+1)\frac{N_w}{N_c}\right] + \|\vec{\nabla}_\perp v\|_2^2 \left[1 - (\rho+\lambda)\frac{N_v}{N_c}\right], \tag{29}$$

so that the sufficient condition (28) implies that one among the two following constraints must at least be satisfied:

$$N_w > N_w^c \equiv \frac{N_c}{(\rho+1)} \quad \text{and/or} \quad N_v > N_v^c \equiv \frac{N_c}{(\rho+\lambda)}. \tag{30}$$

Promoted by incident waves obeying the requirement (28), collapse may therefore take place for a total power N exceeding the critical value $\mathcal{N}_c$:

$$N > \mathcal{N}_c \equiv N_c/(\rho+1) + 4N_c/(\rho+\lambda).$$

Conversely, assuming that N_w and N_v are *below* their respective bounds N_w^c and N_v^c prevents the collapse in the NLS sense, as the gradient norms of the fundamental and second-harmonic waves remain bounded by the finite quantity $H^{(3)}$ in Eq. (29). This situation only applies when $H^{(3)}$ satisfies the condition opposite to Eq. (28), namely, $H^{(3)}>0$. Reexpressing once more the bound (29) in terms of the total power N, it is sufficient to ensure

$$N < \mathcal{N}_{\text{low}} \equiv \min\left\{\frac{N_c}{\rho+1}, \frac{4N_c}{\rho+\lambda}\right\}$$

for preventing the collapse. An alternative way to prove that collapse does not occur in the two-dimensional case for low-power waves is to rewrite the virial relation (14) in the form

$$\partial_z^2 I(z) \geqslant 8\{\|\vec{\nabla}_\perp w\|_2^2[1-(\rho+1)N_w/N_c]$$
$$+ \|\vec{\nabla}_\perp v\|_2^2[1-(\rho+\lambda)N_v/N_c]\}$$
$$\geqslant 2(1-N/\mathcal{N}_{\text{low}})[\|\vec{\nabla}_\perp w\|_2^2 + 4\|\vec{\nabla}_\perp v\|_2^2].$$

We then employ the inequality (19) and reason as in Sec. IV A to conclude that no collapse happens in this case. Originally established for positive coupling constants ρ and λ, this result can be extended to the case $\rho<0$ by repeating the former reasoning on the virial identity now bounded as follows:

$$\partial_z^2 I(z) \geqslant 4[2(\|\vec{\nabla}_\perp w\|_2^2 + \|\vec{\nabla}_\perp v\|_2^2) - (\|w\|_4^4 + \lambda\|v\|_4^4)]$$
$$\geqslant 2(1-N/\alpha)[\|\vec{\nabla}_\perp w\|_2^2 + 4\|\vec{\nabla}_\perp v\|_2^2]$$

for $N<\alpha\equiv\min\{N_c,4N_c/\lambda\}$, after using the Sobolev inequality (25). This absence of collapse thus concerns larger powers than in the preceding situation. For $\lambda<0$, the same kind of treatment can be applied by ignoring the L^4 norm of v in the previous estimate, so that the constraint on the partial powers only concerns N_w.

We can finally observe that, in the so-called one-field limit $v\to 0$ when we disregard Eq. (2), the NLS necessary condition $N_w>N_c$ for a critical collapse is directly recovered from the estimates (29) and (30), since ρ vanishes in this limit, as this coupling constant makes sense as long as the second harmonic wave v has nontrivial values only. In this respect, it can be noticed that the choice $q=1$, in the basic inequality (20) used to get (26), allowed us to derive the sharpest estimates (30) that restore the NLS critical threshold when $v\to 0$ in Eq. (1). This NLS critical power $N_w^c\to N_c$ can also be refound by repeating the above procedure, starting back to the key estimate Eq. (26), which simply vanishes in the case $v=0$.

For the sake of clarity, we henceforth focus our attention on the most salient choice of the coupling parameters promoting the collapse, i.e., from now on, the constants ρ and λ are supposed to be always positive.

C. Competing between the $\chi^{(2)}$ and $\chi^{(3)}$ nonlinearities

Returning to the complete equation set (1) and (2), we investigate the dimensional cases $D=3$ and $D=2$, successively, for arbitrary values of the mismatch parameter β.

1. The supercritical case D=3

We first make use of the inequality (20) with $q=1/2$ on the integral contribution

$$\mathrm{Re}\int(w^2v^*)d\vec{r}_\perp\leqslant\|w\|_4^2\|v\|_2\leqslant\|w\|_4^4+\tfrac{1}{4}\|v\|_2^2, \quad (31)$$

in order to bound the virial identity (15) as follows:

$$\partial_z^2I(z)\leqslant 8\left[H-\left(\beta+\frac{(D-4)}{16}\right)\|v\|_2^2\right]+4(3-D)\|w\|_4^4 + 2(2-D)(\lambda\|v\|_4^4+2\rho\|wv\|_2^2), \quad (32)$$

which leads in the three-dimensional case $D=3$ to

$$\partial_z^2I(z)\leqslant 8[H-(\beta-\tfrac{1}{16})\|v\|_2^2]. \quad (33)$$

A sufficient condition for the collapse to occur is then simply

$$H-(\beta-\tfrac{1}{16})N_v<0. \quad (34)$$

In terms of the conserved quantities N and H, this condition is always ensured whenever H and N satisfy: (i) $H<0$ if $\beta\geqslant 1/16$; (ii) $H<(\beta-1/16)N/4$ if $\beta<1/16$. Condition (i) resembles the sufficient standard condition for collapse, $H<0$, in the pure $\chi^{(3)}$ problem. For $\beta<1/16$, condition (ii) is more restrictive than the latter one and, in particular for $\beta=0$, it appears to be more severe here than in the pure $\chi^{(3)}$ case. This can be explained by the stabilizing effects induced by the $\chi^{(2)}$ nonlinearities that contribute to the virial relation (15), through the quadratically nonlinear potential $\mathrm{Re}\int(w^2v^*)d\vec{r}_\perp$, with a positive quantity which counteracts the requirement $\partial_z^2I(z)<0$ for collapse.

We finally mention that criteria for wave collapse, sharper than Eq. (34), might be found after combining the inequalities (25) and (32) following the steps of the analysis performed in [25] in the context of the NLS equation. However, establishing these criteria would require one to determine the best constants C_w^{best} and C_v^{best} corresponding to Eq. (25) in the 3D case, which is beyond the scope of the present paper.

2. The critical case D=2

Proving the existence of collapsing solutions in the 2D case is not straightforward. Indeed, for $D=2$, the virial relation (15) reduces to

$$\partial_z^2I(z)=8(H-\beta\|v\|_2^2)+4\,\mathrm{Re}\int(w^2v^*)d\vec{r}_\perp, \quad (35)$$

where the condition $(H-\beta\|v\|_2^2)<0$ is not sufficient to assure a blow up and where the potential contribution related to the $\chi^{(2)}$ nonlinearity could even be thought to act as inhibiting the collapse. Even when employing all the inequalities expounded so far, we are not able to bound efficiently this contribution in order to conclude rigorously on the existence of collapse. Instead, we can argue as follows by using the alternative virial formulation (16):

$$\partial_z^2I(z)=4(H-\beta\|v\|_2^2)+4H^{(3)}, \quad (36)$$

where $H^{(3)}$ is still defined by Eq. (23). In view of Eq. (36), a sufficient condition for promoting the collapse should consist in the conjugation of the two constraints:

$$\text{(a)}\ \ H^{(3)}<0, \quad \text{(b)}\ \ H-\beta N_v<0,$$

assuring $\partial_z^2I(z)<0$ if they were satisfied for every z. Using the estimation (29), we easily deduce that condition (a) implies that one of the two inequalities (30), i.e., $N_w(z)>N_w^c$, $N_v(z)>N_v^c$, should be at least verified. For initial data yielding $\mathrm{Re}\int(w^2v^*)d\vec{r}_\perp(z=0)>0$, condition (a) also implies, at least initially, that (b) is systematically satisfied, which surely promotes self-focusing in the early stages of the wave evolution. The problem raised in the present context is that neither $H^{(3)}(z)$ nor $N_v(z)$ are constants of motion, in such a way that we can only expect that the self-focusing process will be favored for initial data with sufficiently high powers. Those are optimized with $N_w(0)>N_w^c$ and $N_v(0)>N_v^c$, that is, $N>\mathcal{N}_c$, as in a purely cubic medium. The condition $N>\mathcal{N}_c$ thus appears to be an *optimal* (or ''*by-excess*'') condition for self-focusing in $\chi^{(2)}-\chi^{(3)}$ media, for which it can be viewed as an upper limit of the collapse threshold. This suggests that, as the quadratic nonlinearities tend to counteract the natural wave dispersion by forming solitons, they should not efficiently arrest the collapse induced by the cubic nonlinearities, whose localizing effects reinforce the quadratic ones. We will later show that, defined under the mass assumptions (30), the Hamiltonian appears to be unbounded from below, from which the possibility of realizing a collapse with the same constraints as in a purely $\chi^{(3)}$ medium follows (see Sec. V A). We will, moreover, confirm the development of collapse under those conditions by means of a variational method elaborated in Sec. VI and numerical simulations presented in Sec. VII. In this respect, it is important to note that the above condition $N > \mathcal{N}_c$ *possibly* assures a collapse dynamics for z close to zero only. It does not necessarily mean that the conditions (a) and (b) ensuring $\partial_z^2 I(z) < 0$ will be fulfilled for every z (even at $z=0$), which will justify why this ''upper'' limit $N > \mathcal{N}_c$ may sometimes be exceeded for some ranges of coupling parameters [see, e.g., Fig. 2(a)].

Furthermore, let us suppose that, *a priori*, the requirements $H^{(3)} < 0$ and $N > \mathcal{N}_c$ are always fulfilled. Then, the condition (b) will be verified whenever the invariants H and N satisfy

$$\text{(i)}\quad H < 0 \quad \text{for } \beta \geqslant 0,$$

$$\text{(ii)}\quad H < \beta N/4 \quad \text{for } \beta < 0,$$

which indicates that the collapse dynamics, and thereby the power threshold for self-focusing, can change according to the value and sign of the mismatch parameter β. In particular, for the initial data under investigation, it will be more difficult to get $H < 0$ for a strictly positive β, than for a zero one, because of the contribution $+\beta N_v$ in H. Consequently, the influence of the mismatch parameter may lead to an increase of the self-focusing threshold when regarding positive values of β, which can alter the basic estimate $N > \mathcal{N}_c$.

On the other hand, we are still able to show that no collapse develops for $D=2$ when the partial powers satisfy conditions opposite to Eq. (30). Indeed, let us bound the virial identity (16) as follows:

$$\partial_z^2 I(z) \geqslant 4\left[\|\vec{\nabla}_\perp w\|_2^2 \left(1 - (\rho+1)\, \frac{N_w}{N_c} \right) + \|\vec{\nabla}_\perp v\|_2^2 \left(1 - (\rho+\lambda)\, \frac{N_v}{N_c} \right) \right] + C \tag{37}$$

by using the Sobolev inequality (25) together with Eq. (26). Here, C is equal to the constant defined in Sec. IV A. It is now clear that if the constraints on the partial powers

$$N_w(z) < N_w^c \equiv N_c/(\rho+1), \quad N_v(z) < N_v^c \equiv N_c/(\rho+\lambda) \tag{38}$$

are *a priori always* satisfied, then we can define $\alpha_p = \min\{N_w^c/N_w, N_v^c/N_v\} > 1$ to find

$$\partial_z^2 I(z) \geqslant (1 - 1/\alpha_p)(\|\vec{\nabla}_\perp w\|_2^2 + 4\|\vec{\nabla}_\perp v\|_2^2) + C \tag{39}$$

and conclude as in Sec. IV A that $I(z)$ can never reach zero in that case. Strictly speaking, we have rather to reason in terms of the total initial datum $N = N_w(0) + 4N_v(0)$: due to the $\chi^{(2)}$ contribution, the partial powers $N_w(z)$ and $N_v(z)$ are not individually conserved along z and, for instance, one of them may increase above its respective critical value for collapse defined by Eq. (38), when both are below initially. Therefore, the estimate (39) must be bounded once more to read

$$\partial_z^2 I(z) \geqslant (1 - N/\mathcal{N}_{\text{low}})(\|\vec{\nabla}_\perp w\|_2^2 + 4\|\vec{\nabla}_\perp v\|_2^2) + C,$$

where the constant

$$\mathcal{N}_{\text{low}} \equiv \min\{N_w^c, 4N_v^c\} \tag{40}$$

denotes the same quantity as the one introduced in Sec. IV B. The reasoning then ends by concluding as in this subsection. This argument shows that, in the critical case $D=2$, choosing incident waves with sufficiently small amplitudes ensuring $N < \mathcal{N}_{\text{low}}$, in a way analogous to the pure $\chi^{(3)}$ problem, prevents the collapse in the presence of both $\chi^{(2)}$ and $\chi^{(3)}$ nonlinearities.

Keeping in mind that, with an interplay of both kinds of nonlinearities, the individual powers N_w and N_v are not conserved separately, we summarize the different regimes of propagation for the coupled waves as follows:

(i) For low powers $N < \mathcal{N}_{\text{low}} \equiv \min\{N_w^c, 4N_v^c\}$, no collapse develops. In this case, the virial integral $I(z)$, which is a measure of the total mean square radius associated with the coupled waves, satisfies $\partial_z^2 I(z) > 0$, so that waves originating from steady-state $[\partial_z I(0) = 0]$ incident data simply spread out with $I(z) \to +\infty$ as $z \to +\infty$.

(ii) For medium-sized powers lying in the intermediate range of values, $\mathcal{N}_{\text{low}} < N < \mathcal{N}_c \equiv N_w^c + 4N_v^c$, we cannot conclude about the fate of the trapped waves. In this ''hybrid'' regime, coupled waves, whose individual powers N_w and N_v remain below their respective critical values $N_w(z) < N_w^c$ and $N_v(z) < N_v^c$ for every z, will surely not collapse at a finite propagation distance. Inversely, self-focusing cannot be excluded *a priori* when the partial norms of the waves satisfy initially one of the two conditions: $N_w(0) > N_w^c$ or $N_v(0) > N_v^c$.

(iii) For high powers $N > \mathcal{N}_c$, collapse of both waves may occur under the requirement $H < 0$ for $\beta \geqslant 0$, or $H < \beta N/4$ for $\beta < 0$. Even though we did not establish rigorous criteria for this occurrence of collapse, we can expect that waves evolving in the presence of both $\chi^{(2)}$ and $\chi^{(3)}$ nonlinearities should logically self-focus under the same requirements as in a purely cubic medium, i.e., with $H^{(3)} < 0$, leading to the optimal condition $N > \mathcal{N}_c$. From the possible unboundedness of the Hamiltonian, this condition, implying that at least one of the two waves possesses a power *above* its associated threshold for self-focusing, can be related to unstable structures. It must be recalled, however, that the expectation of collapse under the constraint $N > \mathcal{N}_c$ may be revised according to the value and sign of the mismatch parameter β. Originally, it follows from a direct comparison with waves evolving in purely $\chi^{(3)}$ media for which the mismatch parameter is zero, so that the former predictions should mainly be relevant for values of β close to zero.

V. SOLITON-TYPE SOLUTIONS

We here investigate the stationary (solitonlike) solutions of the combined $\chi^{(2)} - \chi^{(3)}$ Eqs. (1) and (2). These solutions have the form

$$w(\vec{r}_\perp, z) = w_s(\vec{r}_\perp)\exp(i\Omega z), \quad v(\vec{r}_\perp, z) = v_s(\vec{r}_\perp)\exp(2i\Omega z), \tag{41}$$

where w_s and v_s obey the differential equations

$$-\Omega w_s+\vec{\nabla}_\perp^2 w_s+w_s^* v_s+|w_s|^2 w_s+\rho|v_s|^2 w_s=0, \quad (42)$$

$$-(\beta+4\Omega)v_s+\vec{\nabla}_\perp^2 v_s+\tfrac{1}{2}w_s^2+\lambda|v_s|^2 v_s+\rho|w_s|^2 v_s=0. \quad (43)$$

In the following, we assume the existence of nontrivial solutions $w_s, v_s \neq 0$ and therefore deal with so-called C-type (combined) solitons, by contrast with the so-called V- or W-type solitons investigated in their respective limits $w_s \to 0$ or $v_s \to 0$ in Ref. [18]. We suppose *a priori* a real positive eigenvalue Ω satisfying the requirement $\Omega \geqslant \max\{0,-\beta/4\}$ to assure *localized* soliton-type solutions (bright solitons). We first search for characteristic integral relations satisfied by w_s and v_s: we multiply Eqs. (42) and (43) by w_s^* and v_s^*, respectively, and sum up the real part of the space-integrated results to find

$$\Omega N_s=-(\|\vec{\nabla}_\perp w_s\|_2^2+\|\vec{\nabla}_\perp v_s\|_2^2)-\beta\|v_s\|_2^2 + \frac{3}{2}\,\mathrm{Re}\int (w_s^2 v_s^*)d\vec{r}_\perp+\|w_s\|_4^4+\lambda\|v_s\|_4^4+2\rho\|w_s v_s\|_2^2 \quad (44)$$

with $N_s \equiv \|w_s\|_2^2+4\|v_s\|_2^2$. Next, we multiply Eqs. (42) and (43) by $(\vec{r}_\perp\cdot\vec{\nabla}_\perp w_s^*)$ and $(\vec{r}_\perp\cdot\vec{\nabla}_\perp v_s^*)$, respectively, and integrate the real part of the summed-up resulting equations over space to get

$$\Omega N_s=\left(\frac{2}{D}-1\right)(\|\vec{\nabla}_\perp w_s\|_2^2+\|\vec{\nabla}_\perp v_s\|_2^2)-\beta\|v_s\|_2^2 +\mathrm{Re}\int (w_s^2 v_s^*)d\vec{r}_\perp+\frac{1}{2}\,(\|w_s\|_4^4+\lambda\|v_s\|_4^4 +2\rho\|w_s v_s\|_2^2). \quad (45)$$

Subtracting then Eq. (45) from Eq. (44), we obtain the characteristic relation

$$\mathrm{Re}\int (w_s^2 v_s^*)d\vec{r}_\perp+\|w_s\|_4^4+\lambda\|v_s\|_4^4+2\rho\|w_s v_s\|_2^2 =\frac{4}{D}\,(\|\vec{\nabla}_\perp w_s\|_2^2+\|\vec{\nabla}_\perp v_s\|_2^2), \quad (46)$$

which we employ in order to evaluate the Hamiltonian (10) on the ground-state solutions w_s and v_s:

$$H_s=\frac{2}{D}\,(\|\vec{\nabla}_\perp w_s\|_2^2+\|\vec{\nabla}_\perp v_s\|_2^2)-\Omega N_s. \quad (47)$$

Similarly, we can compute the virial identity on the two-component ground state (w_s, v_s). Using Eqs. (14) and (46), we find the obvious relation $\partial_z^2 I(z)=0$, indicating that the mean square radius associated with the steady-state fundamental and second harmonic waves remains unchanged along z.

The trapped waves, able to converge to the fixed-point solutions of Eqs. (1) and (2), are expected to tend towards such a two-component asymptotic behavior fulfilling $\partial_z^2 I(z)=0$. To investigate the stability of these stationary solitonlike solutions, we employ the following Lyapunov procedure starting with the functional $L \equiv H+\Omega N$. It is well established that, in the framework of NLS-type equations, this functional constitutes an appropriate Lyapunov function, up to some additional constant contributions making it positive [26]. By solving the variational problem $\delta L=0$, one can easily see that, also in the present context, L appears to be a good candidate for analyzing soliton stability, because steady-state solutions to Eqs. (42) and (43) realize an extremum for L. Following Lyapunov's theorem, proving that this extremum consists of a strict minimum should be sufficient for showing the stability of these stationary solutions. To this aim, we will decompose the analysis into two steps. First, we will demonstrate that, for low dimension numbers, the functional L is bounded from below by a function that exhibits a strict minimum, in such a way that L surely admits in turn at least one minimum. Second, we shall solve the variational problem $\delta L=0$ under the constraint of fixed N, in order to identify one minimum of the Lyapunov function and to recover under this constraint that this minimum is reached on the soliton solutions of Eqs. (42) and (43).

Before proceeding, we emphasize that, instead of working with the former function L, it is more convenient to use the functional $S \equiv H-\beta N_v$. Indeed, in view of the above procedure based on the search for bounded integrals and on the constraint of a fixed norm N, making use of S rather than L amounts to obtaining analogous results, since finding a bound from below for S provides a bound from below for L (these two functions satisfy $L \geqslant S$ for localized ground states defined in the eigenvalue domain $\Omega \geqslant \max\{0,-\beta/4\}$). Moreover, working at fixed N, it can be checked that the estimates inferred from L overlap the ones found from S. In this context, it is thus sufficient to use $S \equiv H-\beta N_v$, instead of $L \equiv H+\Omega N$, in order to avoid any redundant treatment when dealing with different values of the mismatch parameter β. Let us notice that reasoning with a fixed N implies that both of the partial norms N_w and N_v are finite in turn, so that the contribution $\beta\|v\|_2^2$ does not here play any crucial role. The functional S is always bounded by $(H-\beta N/4)$ for $\beta>0$ and by H, simply, in the opposite case $\beta \leqslant 0$. Compared with the former functional L, only the absolute value of the minimum is displaced when S is chosen as a Lyapunov function. From these arguments, we easily understand that the most important integral to be evaluated in the present approach is the Hamiltonian: the soliton solutions are then localized stationary points of the Hamiltonian H in the variety of functions with a constant norm N.

To resolve the first point of the above-summarized procedure, we employ the inequality

$$\mathrm{Re}\int (w^2 v^*)d\vec{r}_\perp \leqslant \|w\|_4^2\|v\|_2 \leqslant \sqrt{C_w}N_w^{1-D/4}\|\vec{\nabla}_\perp w\|_2^{D/2}\|v\|_2 \leqslant \frac{\mathcal{C}}{2}\,\|\vec{\nabla}_\perp w\|_2^{D/2}, \quad (48)$$

with $\mathcal{C}=\sqrt{C_w}N^{3/2-D/4}$ and obtain the estimate

$$H-\beta\|v\|_2^2\geqslant\|\vec{\nabla}_\perp w\|_2^2\left[1-\frac{\mathcal{C}}{2}\|\vec{\nabla}_\perp w\|_2^{D/2-2}-\frac{C_w}{2}(\rho+1)N_w^{2-D/2}\|\vec{\nabla}_\perp w\|_2^{D-2}\right]+\|\vec{\nabla}_\perp v\|_2^2\left[1-\frac{C_v}{2}(\lambda+\rho)N_v^{2-D/2}\|\vec{\nabla}_\perp v\|_2^{D-2}\right]. \tag{49}$$

From expression (49), it can be seen that $S\equiv H-\beta\|v\|_2^2$ not only remains bounded from below, but also admits a global minimum for $D<2$ always. In the critical case $D=2$, the functional S is also bounded provided that the partial masses satisfy $N_w<N_c/(\rho+1)$ and $N_v<N_c/(\rho+\lambda)$, separately, that is, for $N<\mathcal{N}_c$. In those cases, the right-hand side of Eq. (49) is bounded parabolically as a function of $\|\vec{\nabla}_\perp v\|_2$, whereas it exhibits a local minimum as a function of $\|\vec{\nabla}_\perp w\|_2$. In the opposite case, when the requirements (30) are fulfilled, i.e., for a total mass $N>\mathcal{N}_c$, S cannot be bounded from below and the mutually trapped waves may collapse under the same conditions as in a purely $\chi^{(3)}$ medium. For $D=3$, Eq. (49) can never be bounded from below, which possibly indicates the instability of any solitonlike structures.

We now resolve the second point of this analysis: we identify one minimum of the functional S together with the fixed-point solutions expected to be stable. First of all, we remark that the stationary solutions of Eqs. (1) and (2) possess, at fixed N, the lowest energy level and must thus realize the minimum of H, or, equivalently, a minimum of S. Since these stationary solutions are assumed to be localized in the transverse plane, they should logically correspond to the above-defined C-type solitons. Let us prove this claim by using the property following which N is fixed and remains invariant under the scaling transformation

$$w(\vec{r}_\perp,z)=\frac{1}{a^{D/2}}\tilde{w}\left(\frac{\vec{r}_\perp}{a},z\right),\quad v(\vec{r}_\perp,z)=\frac{1}{a^{D/2}}\tilde{v}\left(\frac{\vec{r}_\perp}{a},z\right), \tag{50}$$

and employ the parameter a as a Lagrange multiplier for H. This scaling factor a must be identical for the two waves in order to ensure the conservation of the total power N, as resulting from the continuity relations (7) and (8). Introducing Eq. (50) into S transforms it into

$$S_a=\frac{\delta}{a^2}-\frac{\alpha^{(2)}}{a^{D/2}}-\frac{\alpha^{(3)}}{2a^D} \tag{51}$$

with the notations

$$\delta\equiv\|\vec{\nabla}'_\perp\tilde{w}\|_2^2+\|\vec{\nabla}'_\perp\tilde{v}\|_2^2,\quad \alpha^{(2)}=\mathrm{Re}\int(\tilde{w}^2\tilde{v}^*)d\vec{r}_\perp,$$

$$\alpha^{(3)}=\|\tilde{w}\|_4^4+\lambda\|\tilde{v}\|_4^4+2\rho\|\tilde{w}\tilde{v}\|_2^2,$$

where $\vec{\nabla}'_\perp\equiv\vec{\nabla}_{\vec{r}_\perp/a}$. From the expression (51), one deduces that S admits a minimum for $D\leqslant 2$ (given by the minimum of H at fixed N), in accordance with the previous analysis. In addition, this minimum is reached on the solutions satisfying $\partial S_a/\partial a|_{a=1}=0$, i.e., for states verifying the relation

$$\alpha^{(2)}+\alpha^{(3)}=4\delta/D, \tag{52}$$

which is just the relation (46) satisfied by the two-component soliton-solution (w_s,v_s). Consequently, under the constraint of fixed norm N, the minimum of H, as well as the one of the Lyapunov functionals S and L, is reached on this two-soliton family, as expected.

The previous arguments enable us to predict that mutually trapped solitons are stable provided that the space dimension is equal to unity, which thereby provides an analytical confirmation of the stability of the 1D C-type solitons recently observed by Trillo and co-workers [18,20] in numerical computations. Moreover, solitons may be stable in the two-dimensional case, provided that the powers in the incident waves be below their self-focusing thresholds. Note that the present analysis supposes *a priori* that such localized structures exist and are unique for a given class of parameters $(\Omega,\beta,\lambda,\rho)$. It neither proves their existence (numerical observations of 2D solitons will be presented in a forthcoming paper [27]) nor specifies the initial conditions under which they can arise. In fact, the previous conclusions apply to stationary *nonlinear* states created from incident waves that are sufficiently ‘‘massive’’ to generate an attractor able to form solitary waves without promoting their collapse. To illustrate this point, we briefly recall the case of NLS solitons, obtained from Eqs. (1) and (2) by setting $v=0$ in Eq. (1) and ignoring the second equation for v. It is well known that NLS solitons correspond to ‘‘linked’’ states for which the Hamiltonian $H_s=[(D-2)/(4-D)]\Omega N_s$ is negative (see, for instance, [16]). In the case $D=1$, solitons exhibit a sech shape and are stable in the sense that any initial datum, corresponding to a negative-energy state, asymptotically gives rise to a finite set of sech-shaped solitons by evacuating, if necessary, the excess mass through a radiative tail. In a certain sense, formation of stable solitons is permitted, because the collapse process is strictly forbidden in this case. For comparison, solitonlike structures of the 2D NLS equation, which correspond to a zero-energy state $H_s=0$, cannot naturally emerge with a stable shape from a wide class of initial data, because the value $H_s=0$ constitutes a marginal boundary between a continuous wave spreading, ensured for positive-energy states, and a finite-distance blow-up, ensured for negative-energy states. In this situation, the collapse process is permitted. A similar argument enables us to explain the formation of two-component solitons in media with a pure $\chi^{(2)}$ nonlinearity for any dimension $D\leqslant 3$: as a wave collapse is impossible to realize in such media and because steady-state solitary waves are stable, the latter may be formed from a wide range of negative-energy initial data [8].

Thus, generalizing the previous arguments to the interplay of $\chi^{(2)}$ and $\chi^{(3)}$ nonlinearities, we emphasize that the possibility of realizing 2D stable solitonlike structures will depend on the sign of H_s, defined by Eq. (47), and on the values of the individual powers $\|w_s\|_2^2$ and $\|v_s\|_2^2$. If H_s is negative, or N_s far above the critical threshold $\mathcal{N}_c$ for collapse, no stable 2D solitons should be produced because the interval $N_s>\mathcal{N}_c$ overlaps the one associated with solutions which are expected to blow up in this range. In the opposite case $N_s<\mathcal{N}_c$, the creation of two-component solitons is not forbidden *a priori*. However, clearing up this question requires one to identify the localized solutions to the set of differential equations (42) and (43) for a given eigenvalue Ω, and to compute the energy and power integrals attached to these solutions, which is beyond the scope of the present paper.

VI. A VARIATIONAL APPROACH

A. Trial functions describing the coupled waves

In this section, we follow the dynamical (z-dependent) behaviors of localized solutions (w,v) using a variational approach based on self-similar-like substitutions of the form

$$w(\vec{r}_\perp,z)=\frac{1}{\sqrt{A(z)}}\,W\left(\frac{\vec{r}_\perp}{a(z)},z\right),$$

$$v(\vec{r}_\perp,z)=\frac{1}{\sqrt{B(z)}}\,V\left(\frac{\vec{r}_\perp}{b(z)},z\right), \tag{53}$$

where $a(z)$ and $b(z)$ denote the typical z-dependent radius of w and v, respectively. In order to identify the real intensity factors $A^{-1}(z)$ and $B^{-1}(z)$ and the phases of W and V, we insert the solutions (53) into the continuity equations (7) and (8) and obtain

$$\left(\partial_z-\frac{\dot{A}}{A}-\frac{\dot{a}}{a}\,\vec{\xi}_a\cdot\vec{\nabla}_a\right)|W|^2=-\frac{2}{a^2}\,\vec{\nabla}_a\cdot[|W|^2\vec{\nabla}_a\,\arg(W)]-\frac{2}{\sqrt{B}}\,\mathrm{Im}[(W^*)^2V], \tag{54}$$

$$2\left(\partial_z-\frac{\dot{B}}{B}-\frac{\dot{b}}{b}\,\vec{\xi}_b\cdot\vec{\nabla}_b\right)|V|^2=-\frac{2}{b^2}\,\vec{\nabla}_b\cdot[|V|^2\vec{\nabla}_b\,\arg(V)]+\frac{\sqrt{B}}{A}\,\mathrm{Im}[(W^*)^2V], \tag{55}$$

where we have defined $\vec{\nabla}_{\{a,b\}}\equiv\vec{\nabla}_{\vec{\xi}_{\{a,b\}}}$ with $\vec{\xi}_a\equiv\vec{r}_\perp/a(z)$ and $\vec{\xi}_b\equiv\vec{r}_\perp/b(z)$. Here, the dot notation means a differentiation with respect to z. Let us first discuss the self-consistency of the above mass continuity relations with respect to the transformations (53). By "self-consistency," it is meant that the original conservation laws must remain formally unchanged, and therefore be covariant, through these transformations. First, in order to recover a covariant form of these conservation equations, the respective phases of W and V have to expand as follows:

$$\arg(W)=\theta_w(\vec{\xi}_a,z)+\alpha(z)|\vec{\xi}_a|^2,$$

$$\arg(V)=\theta_v(\vec{\xi}_b,z)+\beta(z)|\vec{\xi}_b|^2$$

with $\alpha(z)=a\dot{a}/4$ and $\beta(z)=b\dot{b}/2$, which, in addition, allows us to determine $A(z)=[a(z)]^D$ and $B(z)=[b(z)]^D$ up to some constant factors that can be set equal to unity without loss of generality. Second, keeping the previous results in mind, one then deduces that the exact canceling between the $\chi^{(2)}$ nonlinear contributions required for restoring the total power (9) implies $A(z)=B(z)$, and thus $a(z)=b(z)$. So, from now on, the rescaled variables $\vec{\xi}_a$ and $\vec{\xi}_b$ reduce to the single one: $\vec{\xi}\equiv\vec{r}_\perp/a(z)$ (we will thereby adopt the notation $\vec{\nabla}=\vec{\nabla}_{\vec{\xi}}$). In summary, choosing to model the two coupled waves by

$$w(\vec{r}_\perp,z)=\frac{1}{[a(z)]^{D/2}}\,R_w(\vec{\xi},z)\exp\left[i\,\theta_w(\vec{\xi},z)+i\,\frac{a\dot{a}}{4}\,\xi^2\right], \tag{56}$$

$$v(\vec{r}_\perp,z)=\frac{1}{[a(z)]^{D/2}}\,R_v(\vec{\xi},z)\exp\left[i\,\theta_v(\vec{\xi},z)+i\,\frac{a\dot{a}}{2}\,\xi^2\right], \tag{57}$$

ensures preserving the original structure of the continuity relations whose transformed versions (54) and (55) indeed simplify into

$$[a(z)]^2\partial_z|W|^2=-2\vec{\nabla}\cdot(|W|^2\vec{\nabla}\theta_w)-2[a(z)]^{2-D/2}\mathrm{Im}[(W^*)^2V], \tag{58}$$

$$2[a(z)]^2\partial_z|V|^2=-2\vec{\nabla}\cdot(|V|^2\vec{\nabla}\theta_v)+[a(z)]^{2-D/2}\mathrm{Im}[(W^*)^2V]. \tag{59}$$

Further, we need to make a suitable choice for the functions $(R_w,R_v,\theta_w,\theta_v)$. As it is clear that we cannot find analytically their exact spatial dependences, we postulate that a proper approximation is that the amplitude functions R_w and R_v are exactly self-similar, in the sense that both of them do not explicitly depend on z and therefore reduce to $R_w(\vec{\xi})$ and $R_v(\vec{\xi})$, as is usually assumed in the context of the cubic NLS equation [28–30]. Under this self-similarity condition, the z derivative of the continuity relations (58) and (59) is zero, so that the most natural choice consistent with the vanishing of the right-hand sides of these equations is to impose profile test functions with $\theta_w=\theta_v=0$. Additional phase contributions in the z-dependent forms $\theta_w(z)=\theta_v(z)/2$ could be checked not to alter the coming results. Note that with this self-similar prescription, the partial powers N_w and N_v remain constant along z.

Finally, for modeling the amplitudes R_w and R_v, we first refer to the recent work [31] where Gaussian functions were shown to approach 1D and 2D trapped solitons in $\chi^{(2)}$ media with a great accuracy. On the other hand, we recall that Gaussian trial functions of the form $\exp(-\xi^2/2)$ are reasonably good approximations of sech-soliton solutions to the 1D cubic NLS equation. Furthermore, dynamical solutions constructed with Gaussians of the same form restore the collapsing behaviors of singular solutions to the cubic 2D NLS equation (see, e.g., [29] and [30]). In this case, the critical mass $N_c=11.68$ for a 2D self-focusing is approximated by $N_c\simeq4\pi$ when the true NLS solution is forced with a Gaussian distribution like $\exp(-\xi^2/2)$. In light of this, we therefore apply the following test functions for the coupled waves:

$$w(\vec{r}_\perp,z)=[a(z)]^{-D/2}R_w(\vec{\xi})\exp\left(i\,\frac{a\dot{a}}{4}\,\xi^2\right),$$

$$R_w(\vec{\xi})=\sqrt{P_w}\,\exp\left(-\frac{\xi^2}{2}\right), \tag{60}$$

$$v(\vec{r}_\perp,z)=[a(z)]^{-D/2}R_v(\vec{\xi})\exp\left(i\,\frac{a\dot{a}}{2}\,\xi^2\right),$$

$$R_v(\vec{\xi})=\sqrt{P_v}\,\exp\left(-\frac{\xi^2}{2}\right), \tag{61}$$

where P_w and P_v denote some intensity coefficients to be fixed later on.

B. The dynamical system and related numerical results

We insert the trial solutions (60) and (61) with self-similar amplitudes into the virial identity (14), in order to derive the equation governing the mean radius $a(z)$ of the coupled fundamental and second harmonic waves w and v, namely

$$\ddot{a}=\frac{4}{\mathcal{I}}\left[\frac{\delta}{a^3}-\frac{D}{4}\left(\frac{\alpha^{(2)}}{a^{1+D/2}}+\frac{\alpha^{(3)}}{a^{1+D}}\right)\right] \tag{62}$$

with

$$\mathcal{I}\equiv\|\vec{\xi}R_w\|_2^2+4\|\vec{\xi}R_v\|_2^2,\quad \delta\equiv(\|\vec{\nabla}_\perp R_w\|_2^2+\|\vec{\nabla}_\perp R_v\|_2^2),$$

$$\alpha^{(2)}\equiv\int R_w^2R_v\,d\vec{\xi},\quad \alpha^{(3)}\equiv\|R_w\|_4^4+\lambda\|R_v\|_4^4+2\rho\|R_wR_v\|_2^2.$$

We discuss the dynamical system (62) for each transverse dimension number D and introduce the Gaussian test functions into the above integrals computed in a radially symmetric geometry. By doing so, we find that the equation governing the evolution of the radius $a(z)$ simplifies for any dimension number D into

$$\ddot{a}=\frac{4}{P_w+4P_v}\left[\frac{P_w+P_v}{a^3}-\frac{2^{D-1}P_w\sqrt{P_v}}{6^{D/2}a^{1+D/2}}-\frac{(P_w^2+\lambda P_v^2+2\rho P_wP_v)}{2^{1+D/2}a^{1+D}}\right]. \tag{63}$$

For comparison, the counterpart of Eq. (63) in the context of the cubic NLS equation can easily be recovered by disregarding the contributions in v $(v=P_v=0)$. For instance, in the case $D=2$, the dynamical system describing the evolution of $a(z)$ in the NLS limit $v\rightarrow 0$ is expressed as

$$\ddot{a}=\frac{4}{a^3}\left[1-\frac{N_w}{4\pi}\right] \tag{64}$$

with $N_w=\|R_w\|_2^2=\pi P_w$. It predicts the occurrence of a 2D (critical) collapse at a finite propagation distance z_c with a vanishing $a(z)$. This collapse develops for incident steady-state $[\dot{a}(0)=0]$ Gaussian beams, whenever the power N_w exceeds the critical threshold $N_c\simeq 4\pi$. In the opposite case $N_w<N_c$, the wave spreads out with a diverging $a(z)$.

In solutions to the two-component system (1) and (2), collapse is expected to occur in purely $\chi^{(3)}$ media if the sufficient condition $H^{(3)}<0$ is satisfied by the incident waves (see Sec. IV B), while we cannot definitively conclude on the occurrence of collapse for waves propagating in the presence of competing quadratic and cubic nonlinearities. In the latter context and in view of the virial expressions (35) and (36), we impose *a priori* this sufficient condition reading for a nonzero β:

$$H(z)-\beta\|v\|_2^2=\frac{\delta}{[a(z)]^2}-\frac{\alpha^{(2)}}{[a(z)]^{D/2}}-\frac{\alpha^{(3)}}{2[a(z)]^D}<0. \tag{65}$$

This quantity enters the first integral of motion associated with the dynamical system (62):

$$G(z)=\frac{\mathcal{I}}{4}[\dot{a}(z)]^2+[H(z)-\beta N_v]=G(0). \tag{66}$$

Beginning with the case $D=1$, we observe from Eqs. (63) and (66) that not only a collapse cannot be realized in that case with $\delta>0$, but also the dynamical system (62) admits fixed points corresponding to stable positions attained at the minima of the functional (65). These fixed points indicate the asymptotic formation of one-dimensional C-type solitons that are stable, following the discussion in Section V. The wave radius $a(z)$, given by the variational analysis, oscillates with a constant frequency, which represents steady oscillations around the soliton form. These oscillations mean that if an excess of mass could have been radiated away, solitons would have formed.

In the situation $D=2$, it is clear that, since the integral contribution $\alpha^{(2)}$ is positive, the condition (65) allowing collapse will surely be fulfilled if $2\delta-\alpha^{(3)}<0$, i.e., when P_w and P_v verify the inequality

$$\Omega(P_w,P_v)\equiv P_w[4-(\rho+1)P_w]+P_v[4-(\rho+\lambda)P_v]+\rho(P_w-P_v)^2<0, \tag{67}$$

implying hence $P_w>P_w^c\equiv 4/(\rho+1)$ and $P_v>P_v^c\equiv 4/(\rho+\lambda)$ with $\rho,\lambda>0$. These constraints then recover the conditions (30) and $N>\mathcal{N}_c$ for collapse in purely $\chi^{(3)}$ media, keeping in mind the relations $N_{w,v}=\pi P_{w,v}$ and $N_c=4\pi$. In those conditions, the system (63), rewritten as

$$\ddot{a}=\frac{1}{P_w+4P_v}\left[\frac{\Omega(P_w,P_v)}{a^3}-\frac{4P_w\sqrt{P_v}}{3a^2}\right], \tag{68}$$

shows that, starting from steady-state waves having a normalized radius $[a(0)=1,\ \dot{a}(0)=0]$, a collapse at a finite distance z_c will always take place with $\Omega(P_w,P_v)<0$. Inversely, if we now consider the inequality opposite to Eq. (67), which is satisfied with $P_w<P_w^c$ and $P_v<P_v^c$, Eq. (68) may contain a center-type equilibrium position corresponding to the minimum of (65), for intensities rather close to their self-focusing thresholds. This indicates the possible formation of two-dimensional C-type solitons. For intensity values far below the self-focusing thresholds, the waves disperse with $a(z)\rightarrow+\infty$ as z increases, similarly to the dispersing NLS solutions. These behaviors are consistent with the points (i)–(iii) emphasized at the end of Sec. IV.

Besides, three-dimensional waveforms can be seen to collapse rapidly when P_w and P_v are sufficiently large to ensure $\ddot{a}(0)<0$ initially. More precisely, collapse surely occurs when the condition (65), supplemented by $N_v/16=\pi^{3/2}P_v/16$

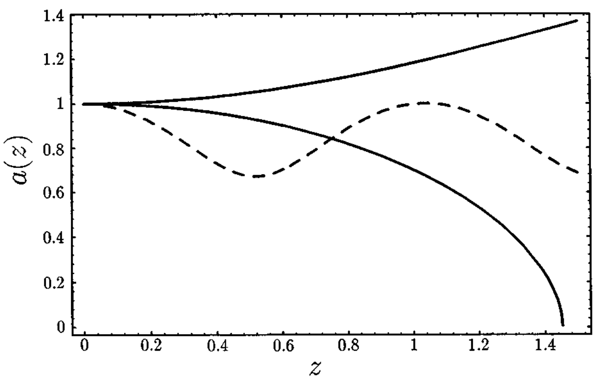

FIG. 1. Evolution of the radius $a(z)$ vs z integrated from Eq. (63) for $D=1$ and $P_w=P_v\equiv P=1$ (dashed line). In this case only, the z axis has been rescaled by a factor $\sqrt{10}$ ($z_{\rm true}=\sqrt{10}z_{\rm Fig.}$) to see clearly the wave radius oscillates around a stable soliton-type fixed-point solution. Solid lines represent the radius $a(z)$ computed in the case $D=2$ for the intensity factors $P=1.5>P_c$ (lower curve), yielding a collapse at a finite propagation distance, and $P=0.8<P_c$ (upper curve), implying a wave spreading.

to recover the sufficient requirement (34), is initially satisfied with $a(0)=1$. In the opposite situation, the waves generally spread out, like the standard dispersing solutions of the cubic NLS equation. We can therefore conclude from these approximated behaviors that no stable soliton should be formed in combined $\chi^{(2)}-\chi^{(3)}$ materials when $D=3$, in agreement with the results of Sec. V. This conclusion is also consistent with the property according to which the functional (65) contains no local minimum in that case.

Typical behaviors of the wave radius $a(z)$ have been illustrated for $D=1$ and $D=2$ in Fig. 1, obtained by a numerical integration of the variational equation (63). We have chosen $\rho=2$ and $\lambda=1$, so that the critical partial powers have the identical values $P_w^c=P_v^c=4/3$. For the sake of simplicity, we have considered waves with equal values for their intensities $P_w=P_v\equiv P$ and characterized by a normalized incident radius $a(0)=1$ with $\dot{a}(0)=0$. Behaviors comparable to the present ones would be obtained by considering incident waves of different amplitudes. In the case $D=1$ plotted with a dashed line, we observe that the radius $a(z)$ oscillates with a constant frequency, which represents steady oscillations around the soliton form for the incident intensity factor $P=1$. Besides, described with a radius plotted in solid lines, 2D waveforms are observed to collapse for $P=1.5$, and to spread out for $P=0.8$. More generally, collapse takes place when P exceeds the critical threshold $P_c=4/3$ for self-focusing, as expected, whereas the waves continuously disperse in the opposite range $P<0.9<P_c$. These results will later be compared with a direct numerical integration of Eqs. (1) and (2) detailed in the next section. Finally, we mention that 3D waves were found to exhibit a radius $a(z)$ tending to zero at a finite distance z_c for $P\geqslant 1.5$ and diverging for lower intensity values, which describes a finite-distance collapse and an asymptotic spreading of the waves, respectively, without forming soliton-type structures.

We must stress here that, by virtue of the self-similarity assumption imposed in the variational model, the estimates of N_w and N_v remain unchanged along z, in such a way that the mass exchanges between the fundamental and second harmonic waves are strictly forbidden by this approximation method. Moreover, the present variational analysis assumes waveforms that remain forced with a Gaussian shape. Consequently, these constraints may hereby introduce some discrepancies (in the collapse distance for instance), compared with the true numerically integrated solutions to Eqs. (1) and (2). As an example, it can be seen from the relation (65) that the Hamiltonian—that is a constant of motion—diverges in the collapse regimes $a(z)\to 0$ for $D=3$, when it is approximated with the self-similar trial solutions (60) and (61). In the present context, the variational approach thus gives approximated results that, although restoring the general tendencies of the coupled waves, must be regarded with caution.

VII. NUMERICAL COMPUTATIONS

In this section, we present the results issued from numerical integrations of Eqs. (1) and (2) with $s=+1$, $\beta=0$, and $\rho,\lambda>0$. We use a split-step Fourier method of second order in z (see, e.g., [32]), solving the linear and nonlinear parts of the equations separately. The linear part is solved using Fourier transformations, while the nonlinear part is solved with a fourth-order Runge-Kutta method. The conservation of the total power N was checked throughout the integration, with a relative error of less than 10^{-5} for all the results presented.

Collapsing solutions are investigated using a fixed resolution in all the coordinates. Thereby, two things can happen: (1) If the step size in the direction of propagation, Δz, is too coarse, then the total power N will not remain conserved when the collapse singularity is approached. Because we define the maximum allowable deviation $\Delta N=10^{-5}$, this means that the program will simply have to stop at a certain distance z_c preceding the true singularity. If we chose Δz sufficiently small, so that N would remain conserved without exceeding the allowable deviation in the absence of collapse, then z_c could be expected to be a good measure of the ''real'' collapse distance. (2) If the resolution in the transverse direction, $\Delta x=\Delta y$, is too coarse, then N will be conserved for all z, but at some distance close to the collapse, the system will behave as if it was discrete and we will observe what is known as trapping [33]. Following this procedure, a collapse distance could still be estimated by visual inspection, after the program integrated out to the preset integration length.

In the following, we have chosen the resolution in the transverse direction to be sufficiently fine, so that the scheme (1) applies: for instance, in the 2D case, integrations were performed with periodic boundary conditions within a box containing $N_x\times N_y$ points for transversal integration step sizes $\Delta x=\Delta y=0.1$ covering the simulation domains $L_x=N_x\times\Delta x$ and $L_y=N_y\times\Delta y$. In the 1D case, a better resolution was allowed. From this setup, we thus define the collapse distance as the distance where the deviation in N exceeds $\Delta N=10^{-5}$, and the program stops the integration automatically. By doing so, we do not have to estimate an upper limit of the possible collapse distance in order to limit the length of integration, as we should do with the scheme (2). In particular, when calculating the curves depicted in Figs. 2 and 3, this procedure offers a considerable reduction in computer time.

All the numerical computations have been performed for

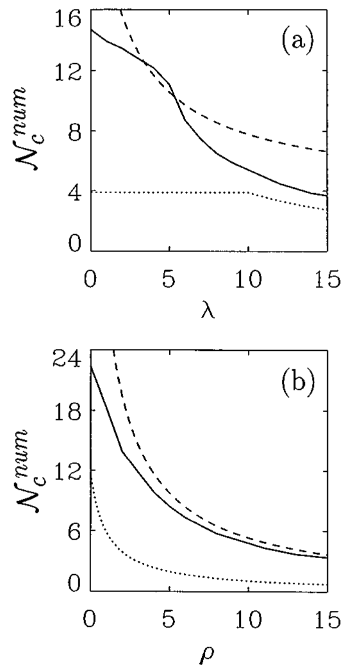

FIG. 2. Critical power $\mathcal{N}_c^{\mathrm{num}}$ above which both waves w and v are numerically observed to collapse for different values of the coupling parameters: ρ=2 with varying λ (increment step Δλ=1) (a), λ=1 with varying ρ ($\Delta\rho$=1) (b). The dashed line represents the theoretical value of the collapse threshold $\mathcal{N}_c$ and the dotted one the lowest bound $\mathcal{N}_{\mathrm{low}}$. The lengths of the integration box used for solving Eqs. (1) and (2) are $L_x = L_y = 12.8$, $L_z = 8$ for a grid with $N_x \times N_y = (128)^2$ points and an integration step size $\Delta z = 10^{-3}$ in the z direction.

incident waves having the same Gaussian shapes as the ones used in the variational approach, namely,

$$w(r_\perp,0) = v(r_\perp,0) = \sqrt{P}\ \exp(-r_\perp^2/2), \qquad (69)$$

with an identical intensity factor P and an identical initial wave half-width $a(0)=1$. The mismatch parameter β was set equal to zero. The specifications on the chosen lengths of

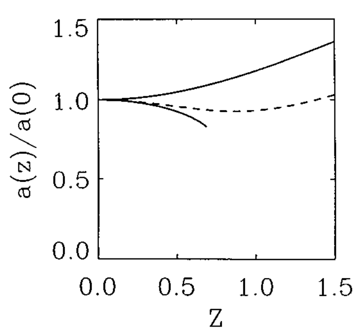

FIG. 3. Normalized mean wave width $a(z)/a(0)$ vs the propagation distance z, computed from the ratio of the virial integrals $I(z)/I(0)$ with the initial data (69) for different dimension numbers D and different intensity coefficients P: $D=1$, $P=1$ (dashed line) with the simulation parameters: $L_x = 51.2$, $N_x = 4096$, and $\Delta z = 10^{-3}$; $D=2$, $P=1.5$ (solid line, lower curve) and $D=2$, $P=0.8$ (solid line, upper curve) with $L_x = L_y = 25.6$, $N_x = N_y = 256$ for an integration step size $\Delta z = 10^{-3}$.

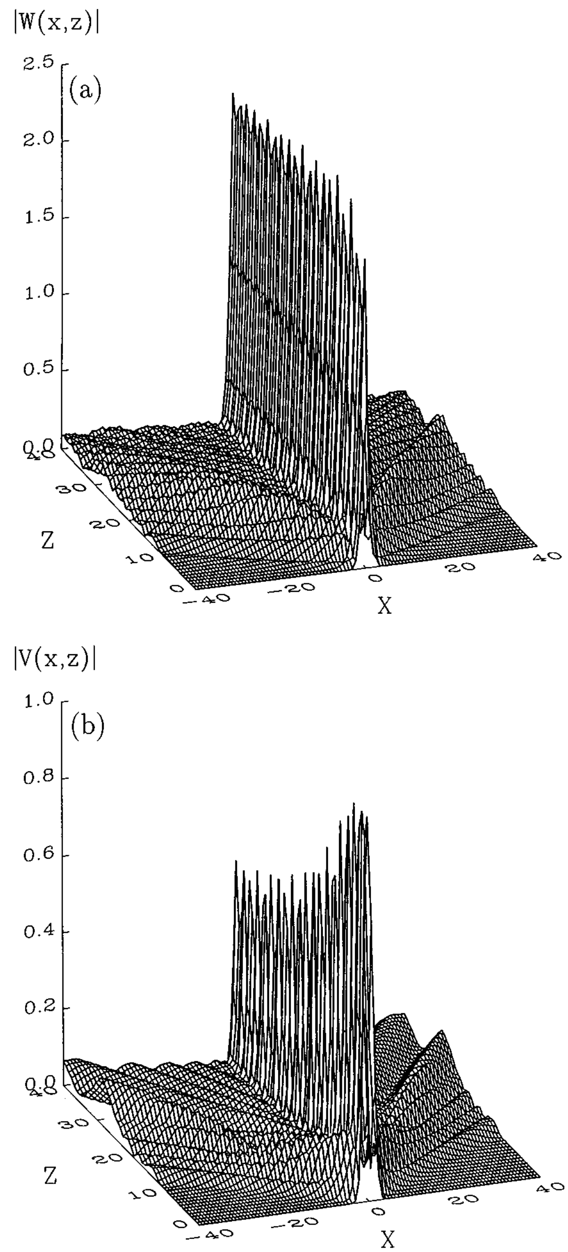

FIG. 4. Evolution of 1D wave amplitudes $|w(x,z)|$ (a) and $|v(x,z)|$ (b) vs z towards long-living solitonlike structures for an incident intensity $P=1$. The parameters of the numerical simulation are $L_x = 204.8$, $N_x = 8192$, and $\Delta z = 10^{-4}$.

integration L_x, L_y, the grid sizes N_x, N_y and the integration step size along z have been pointed out in the figures. The numerically revealed value of the critical total power engaged in the collapse event in the case $D=2$ has been plotted in Figs. 2(a) and 2(b) for different coupling parameters ρ and λ by varying P. This value, denoted by $\mathcal{N}_c^{\mathrm{num}}$, lies between the two theoretical limits $\mathcal{N}_c$ and $\mathcal{N}_{\mathrm{low}}$, as expected. From Fig. 2(a), it can be noted, however, that $\mathcal{N}_c^{\mathrm{num}}$ slightly exceeds the upper threshold for collapse $\mathcal{N}_c$ in some finite range of λ for a fixed ρ=2. This weak discrepancy can be explained by the fact that the constraint $N > \mathcal{N}_c$ does not necessarily mean that the requirements for collapse, as $H<0$ and $H^{(3)}(z)<0$ in the case β=0, are fulfilled whatever z may be for this interval of parameter values.

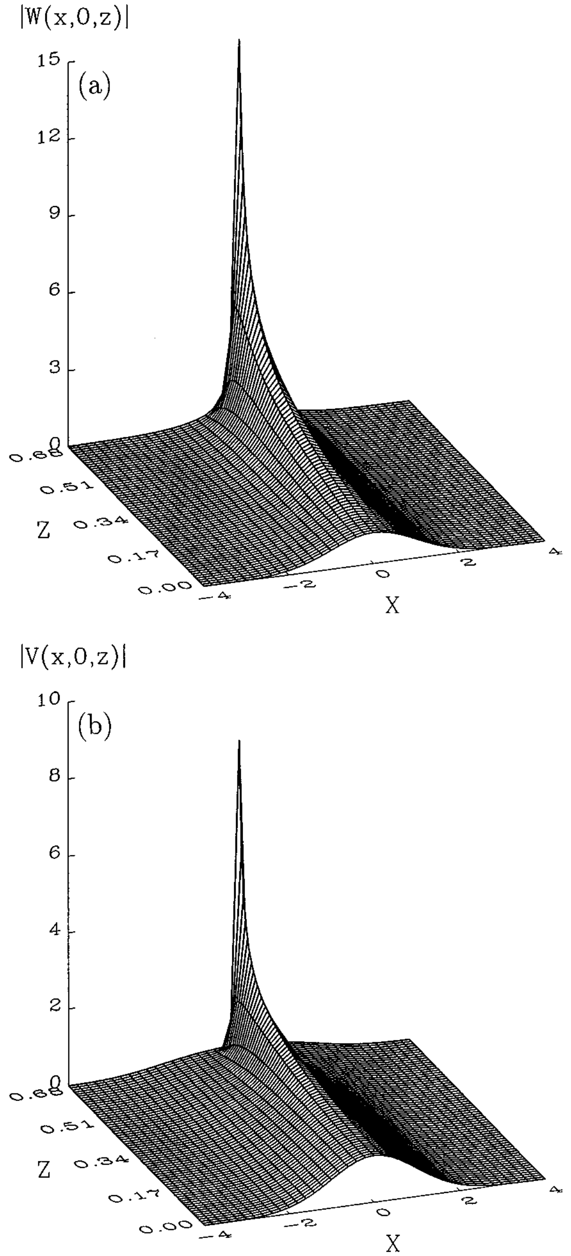

FIG. 5. Collapse of the coupled wave amplitudes $|w(x,0,z)|$ (a) and $|v(x,0,z)|$ (b) vs the propagation distance z, plotted along the line $y=0$ in the case $D=2$ for an intensity factor $P=1.5$. The simulation parameters are $L_x=L_y=25.6$, $N_x=N_y=256$, and $\Delta z=10^{-3}$.

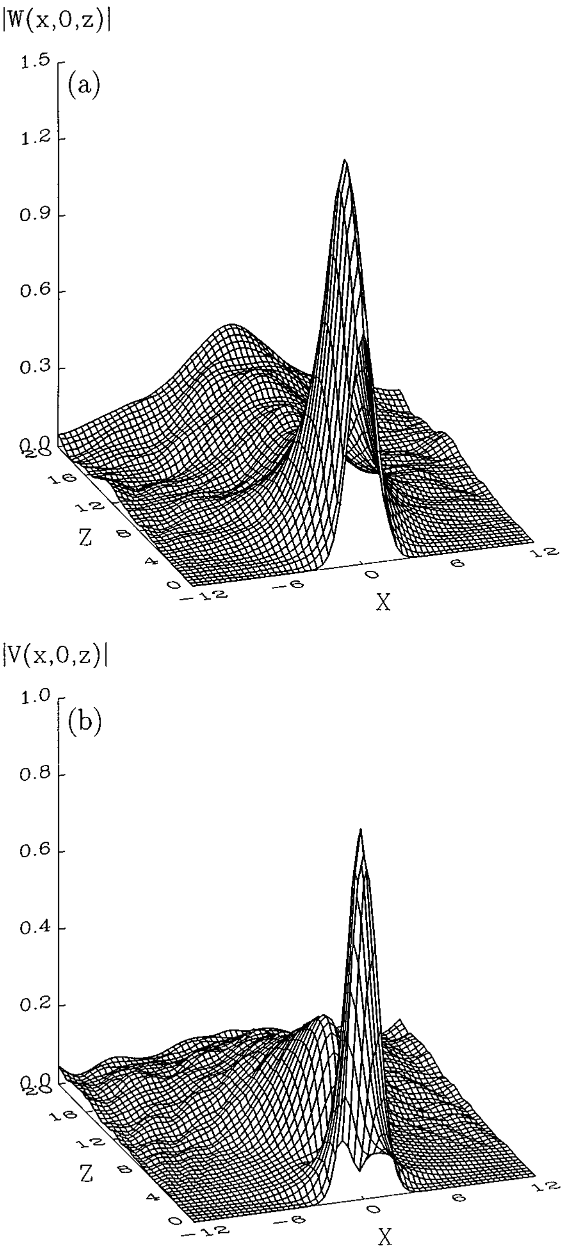

FIG. 6. Spreading of the waves w (a) and v (b) vs z in the 2D case for a weak intensity factor $P=0.8$. The simulation has been performed with the same parameters as the ones indicated in Fig. 5.

In the following figures, the values $\rho=2$ and $\lambda=1$ have been used for comparison with the results obtained from the variational approach. For this choice of coupling parameters, collapse of both waves was observed to occur for $D=2$ whenever $P>0.885$ and the two waves were observed to spread out in the opposite domain of P values: $P<0.885$. These observations are in a good enough agreement with the approximate behaviors resulting from the variational model, according to which collapse develops for $P>P_c=4/3$ while waves can be expected to disperse for $P<0.9$.

Figure 3 illustrates the total mean radius $a(z)$ of the waves, computed from the virial integrals and normalized with respect to $a(0)$. In the one-dimensional case, the radius $a(z)$ plotted with a dashed line for $P=1$ first decreases, then forms a minimum, and afterwards starts to increase asymptotically with z. This increase can be explained by the evacuation of the excess mass to the boundaries, which allows the trapped waves to relax to soliton shapes. In its variational counterpart plotted in Fig. 1, the radius $a(z)$ can be seen to exhibit a similar decrease in the early stage of propagation. However, unlike the true wave radius plotted in Fig. 3, $a(z)$ afterwards describes steady-state oscillations around the soliton solution. This discrepancy results from the constraint of conserved partial masses for self-similar trial functions, which cannot restore the evacuation of the mass excess. Due to this limitation of the variational approach, the wave radius depicted in Fig. 1 cannot reach, e.g., a steady-state value that should have corresponded to the mean soliton width. The upper solid-lined curve plotted in Fig. 3 represents a continuously increasing radius $a(z)$ associated with a wave spreading in the two-dimensional case for a weak intensity factor: $P=0.8$. It can be seen to be in excellent agreement with the behavior illustrated in Fig. 1 for the same data. The lower curve indicates a continuous decrease of $a(z)$ when both waves collapse in the 2D case for an intensity coefficient $P=1.5$ exceeding the critical threshold $P_c=4/3$. Note that the singularity develops before the half-width $a(z)$ reaches zero. Besides the above-recalled limitations of the numerical scheme that partly prevent the solutions from reaching exactly the collapse distance z_c, this arrest in the vanishing of $a(z)$ can also be explained by the well-known property according to which collapsing solutions of NLS-type equations generally blow up and cease to exist before their associated virial integral vanishes. In spite of these discrepancies related to the inner structure of the singular solutions to Eqs. (1) and (2), we can conclude that the variational approach restores with reasonably good accuracy the main behaviors, i.e., spreading and collapse, characterizing the evolution of the wave envelopes w and v.

In Fig. 4, the amplitudes of the coupled waves w and v have been plotted in the one-dimensional case for $P=1$. They illustrate the evolution of 1D trapped waves towards stable solitonlike waveforms. During the early stages of propagation, the excess of mass outgoing from the core of both waves to the boundaries is radiated away. Figures 5 and 6 show in the case $D=2$ the formation of self-focusing spikes, due to collapse, for the intensity factor $P=1.5$, and the spreading of both waves when they carry a lower power with $P=0.8$, respectively. The complete dispersion of the envelope v, shown in Fig. 6(b), should force w to vanish in turn at large z, due to the mutual coupling of the two waves described by Eqs. (1) and (2).

VIII. CONCLUSION

After deriving the main invariants and the virial identity describing the evolution of the mean square radius, we have investigated the mathematical properties of the coupled fundamental and second-harmonic waves propagating in an optical medium with both quadratic and cubic (Kerr) nonlinearities. Apart from few particular results concerning different coupling parameters, this study was mainly devoted to so-called attractive nonlinear potentials, for which the coupling constants ρ and λ are both assumed to be positive. In addition, the dispersion coefficient s was set equal to $+1$.

Summarizing the principal results obtained here, we have shown that 3D coupled waves can self-focus and collapse at a finite propagation distance z_c under the sufficient requirements (34), depending on the Hamiltonian and the total power in the incident waves, as well as on the parameter β, which represents the relative strength of the quadratic and cubic nonlinearities. In this situation, the absence of local minima in the functional dependences of the Hamiltonian and the unboundedness of the latter indicate that no stable solitonlike states can emerge from the mutually trapped waves.

On the contrary, in the simpler basic model of 1D trapped waves, collapse has been proved to never occur. In this case, the waves can remain coupled and asymptotically behave as self-trapped solitons that have been shown to be stable.

Finally, for the case $D=2$, although no exact criterion for the existence of collapsing solutions has been established, we have proven that collapse does not occur when the powers in the two waves remain below some critical values for all z, namely, $N_w(z)<N_w^c=N_c/(\rho+1)$ and $N_v(z)<N_v^c=N_c/(\rho+\lambda)$, which is assured when the total power is sufficiently low and satisfies $N<\mathcal{N}_{\text{low}}\equiv\min\{N_w^c,4N_v^c\}$. These conditions can be viewed as the opposite ones to the requirements (30) that assure, in the two-dimensional case, the collapse of both wave envelopes when the latter evolve within a purely $\chi^{(3)}$ medium. On the basis of the virial expression (36) and by means of a variational approach, we also displayed strong evidence, supported by a numerical confirmation performed for $\beta=0$, that the mutually trapped waves can undergo a collapse under these same requirements (30), even though the presence of $\chi^{(2)}$ nonlinearities is in favor of stabilizing the coupled waves. Imposing both the constraints (30) implies that the initial total power N carried by the incident waves on the whole has to exceed the critical value $\mathcal{N}_c=N_w^c+4N_v^c$. The most salient role played by the quadratic nonlinearities alone is, in fact, that they may counteract the natural spreading of the waves by localizing the latter and force them to remain mutually trapped, so that the fundamental and second harmonic waves can evolve under the form of stable coupled solitons. However, this behavior does not exclude the feasibility of a wave collapse in media being also sensitive to the wave intensity, i.e., to the Kerr effect. The $\chi^{(3)}$ nonlinearity can thus dominate not only the natural dispersion of the waves, but also the stabilizing influence of the $\chi^{(2)}$ nonlinearity, which justifies the collapse of both waves. Nevertheless, as the self-focusing power threshold may vary with the mismatch parameter in combined $\chi^{(2)}-\chi^{(3)}$ media, this direct comparison with a purely cubic medium, for which β vanishes, restricts the validity of our theoretical predictions for collapse to values of β close to zero.

Finally, we postpone to forthcoming papers the question of realizing stable two-dimensional solitons in the presence of both quadratic and cubic nonlinearities. In addition, we underline that the role of large mismatch parameters on the collapse dynamics, the possible mass exchanges between the fundamental and second-harmonic waves and their influence on the individual mean square radius of the coupled waves, have been disregarded in the present study, whose aim was mainly to give global dynamical behaviors. These problems should also be cleared up in the near future.

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